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# ON THE THERMOPLASTICITY CONSTITUTIVE RELATIONS FOR ISOPTOPIC AND TRANSVERSELY ISOTROPIC MATERIALS

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***ABSTRACT***

***The widely used engineering construction materials such as fiber and laminated composite materials are usually under the thermomechanical forces and undergoes thermoplastic deformations. These composites may be considered as a transversely isotropic or orthotropic materials. In this paper, the plasticity constitutive relations for isotropic and transversely isotropy materials proposed in [33]are developed taking into account the temperature and written up the strain and stress space thermoplasticity constitutive relations for aforementioned materials. For simplicity, thermoplasticity theories are restricted to a small deformations. The usefulness and privileges of the strain space thermoplasticity constitutive relations for the formulation the coupled thermomechanical boundary value problems are discussed. It is found that the strain space thermoplasticity constitutive relations are more convenient for numerical solution of the coupled thermoplasticity boundary value problems as compared to stress space theory.***

***KEYWORDS Constitutive Relations, Thermoplasticity Theory, Strain-Stress Space, Isotropy, Transversely Isotropic & Coupled Problems***

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## 1. INTRODUCTION

Despite the advances made in the theory of plasticity and thermoplasticity for small and large deformations, the construction of adequate constitutive relations for isotropic and anisotropic materials remains relevant. The investigation of the plastic deformations of transversely isotropic materials under the thermomechanical forces is important in analyzing the material structures. It is well known that fiber and laminated composites may be considered as a transversely isotropic or orthotropic materials. In recent years, in conjunction with the development of composite materials, researchers have proposed various types of plasticity theories for fiber reinforced, laminated and other composite materials. Aboudi [1] proposed the thermomechanical continuum theory for the prediction of the average behavior of unidirectional fiber reinforced graphite-aluminum composite under various types of mechanical and thermal changes. It should be noted that the theory of plasticity for orthotropic bodies was first proposed by Hill [2]. For fiber reinforced composites, Dvorak and Bahei-EL-Din [3] proposed the constitutive relation for transversely isotropic materials. Constitutive relations of inelasticity for anisotropic materials based on tensor function representations was considered in Boehler and Sawczuk [4], Murakami and Sawczuk [5] and Pobedria [6]. In the literature, there are many works on the thermoplasticity constitutive relations for isotropic, anisotropic and

fiber reinforced composite materials; see [6-13]. Kawai et al. [8] considered a phenomenological thermoviscoplastic model for investigating the stress-strain field in carbon/epoxy composites at high temperatures. Thermoplasticity theories at finite strains are proposed by Green and Naghdi [14], Casey [15] and Miehe [16]. Ulz [17] used a Green-Naghdi's approach [14] for modeling the anisotropic thermoplasticity constitutive relations in the logarithmic Lagrangian strain-entropy space at finite strains. Thermoplastic constitutive laws for concrete and rocks are considered in[10] and [18]. A comprehensive review for thermoplasticity theory at finite strains can be found in Naghdi [19].

Investigation of the joint influence of the thermomechanical forces on the deformation process of materials is an actual problem of solid mechanics and is usually called the coupled problem of the thermoelasticity or thermoplasticity. In investigating the coupled thermoelasticity problems, Biot [20], Lord and Shulman [21], Youssef [22] introduced a generalized coupled theory with a wave-type heat equation. The coupled thermoplasticity problems are considered in [9, 23-28]. The thermomechanical coupling problems, in which the mechanical response of the structure depends upon its thermal behavior and vice-versa is considered by Sloderback and Pajak [24]. Vaz Jr. et al. [25] modelled the coupled effects between ductile damage and temperature evolution. Thermomechanical coupled boundary value problems are solved by many [23, 26-29].

The strain space formulation for plasticity theory was proposed by Naghdi and Trapp [30], Casey and Naghdi [31], and they showed that the plastic strain rate is normal to the loading surface, whereas Yoder and Iwan [32] considered an alternative associated flow law using the so called a stress relaxation tensor $d\sigma_{ij}^{p}$ which is normal to the loading surface in strain space. Note that the stress relaxation tensor differs from the plastic strain $d\varepsilon_{kl}^{p}$ with the elasticity constants $C_{ijkl}$[32]. The acceptance of the stress relaxation tensor $d\sigma_{ij}^{p}$ allows to derive more simple constitutive relations depending on plasticity functions $\varepsilon = \varepsilon(\sigma^{p})$ constructed on the base of the experimental deformation diagram. Note that, in case of classical stress space theories the plasticity functions are found based on the relation $\sigma = \sigma(\varepsilon^{p})$. Nik Long et al. [33], in the framework of the strain space formulation of plasticity, proposed the plasticity constitutive relations for isotropic and transversely isotropic materials, and compared with those of the classical ones. The introduction of the stress relaxation tensor $d\sigma_{ij}^{p}$ allows researchers to note the symmetry of all scalars and tensor variables of the stress and strain spaces in the formulations of plasticity. Casey and Naghdi showed that the stress and strain space formulations of plasticity [34] are not equivalent. In [13, 36, 37] the strain space formulation were used for the construction of the plasticity and thermoplasticity constitutive relations for anisotropic materials.

In the present work, the thermoplasticity constitutive relations are developed for isotropic and transversely isotropic materials, which are a generalization of the theory of plasticity considered in Nik Long et al. taking into account the temperature. The strain and stress space thermoplasticity constitutive relations for isotropic and transversely isotropic materials are constructed. For simplicity a small deformation is considered, and proposed that the elastic constants do not depend on temperature. It is shown that the strain space formulation is superior to the traditional stress space.

In Section 2 using the alternative and classic associated flow laws of plasticity the strain and stress space thermoplastic constitutive relations for isotropic materials are discussed. Taking the loading functions in the Mises's form and representing the hardening functions(thermoplastic surfaces), a comparison of the strain and stress space constitutive relations is carried out. In Sections 3 and 4, based on the alternative and classic associated flow laws considered in previous section and using the invariant theory of tensor functions, the strain and stress space thermoplasticity constitutive

relations for transversely isotropic materials are developed.

## 2. ON THE PLASTICITY AND THERMOPLASTICITY CONSTITUTIVE RELATIONS

Usually in the theory of plasticity the loading surfaces are considered in stress space. However, it is well known that the stress space plasticity theories lead to the unreliable results in the regions of material behavior such as those that correspond to the flat and falling portions of the typical engineering stress-strain curve for uniaxial tension of ductile metals [19]. In order to overcome these difficulties, Naghdi and Trapp [30] and Casey and Naghdi [31] proposed a strain-space plasticity theory, for which the loading surface is considered in the strain space, and the plastic strain is orthogonal to the loading surface. Yoder and Iwan [32], instead of plastic strain $d\varepsilon_{kl}^{p}$, introduced the so called stress relaxation tensor $d\sigma_{ij}^{p}$ which is obtained by multiplying the plastic strain to material coefficients $C_{ijkl}$ i. e. $d\sigma_{ij}^{p} = C_{ijkl} d\varepsilon_{kl}^{p}$. According to Yoder and Iwan the stress relaxation tensor $d\sigma_{ij}^{p}$ can be found from the alternative associated flow law

$$d\sigma_{ij}^{p} = d\lambda \frac{\partial F}{\partial \varepsilon_{ij}} \tag{1}$$

where $d\lambda$ is a positive differential parameter, $F$ is a loading function in strain space. This is not contradict to the strain space formulation considered in Naghdi and Trapp [31]. The strain space formulation of plasticity based on the stress relaxation tensor allows them to construct a constitutive relations and loading criterions depending on strains and strain increments. Let the loading function in strain space, taking into account the temperature $T$ be defined as

$$F(\varepsilon_{ij}, T, \omega) = 0, \tag{2}$$

where $\varepsilon_{ij}$ is a strain tensor. The loading function depends on plastic strain $\varepsilon_{ij}^{p}$ in a implicit form through the parameter

$$\omega = \int \varepsilon_{ij} d\sigma_{ij}^{p}, \qquad d\sigma_{ij}^{p} = C_{ijkl} d\varepsilon_{kl}^{p}. \tag{3}$$

As shown in Nik Long et al. [33] and Chen and Han [34] the parameter $\omega$ is symmetric to $\chi = \int \sigma_{ij} d\varepsilon_{ij}^{p}$ and it describes the work done during the plastic deformations in closed cycles on strain [38] and stresses [39]. Differentiate Eq. (2) and taking into account the expressions (3) and (1) yields

$$d\lambda = H\left(\frac{\partial F}{\partial \varepsilon_{ij}} d\varepsilon_{ij} + \frac{\partial F}{\partial T} dT\right) \tag{4}$$

Where

$$H = -\left(\frac{\partial F}{\partial \omega} \varepsilon_{ij} \frac{\partial F}{\partial \varepsilon_{ij}}\right)^{-1}. \tag{5}$$

It is known that the constitutive relation for plasticity may be written in the form [32, 33]

$$d\sigma_{ij} = C_{ijkl} d\varepsilon_{kl} - d\sigma_{ij}^{p}. \tag{6}$$

Substituting (1) into (6), and taking into account Eq.(4) and the thermoelastic deformations $d\varepsilon_{kl}^{T} = \alpha \delta_{kl} dT$ we obtain

$$d\sigma_{ij} = C_{ijkl}(d\varepsilon_{kl} - \alpha \delta_{kl} dT) - H\left(\frac{\partial F}{\partial \varepsilon_{kl}} d\varepsilon_{kl} + \frac{\partial F}{\partial T} dT\right) \frac{\partial F}{\partial \varepsilon_{ij}} \tag{7}$$

where $C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$, $\lambda,\ \mu$ are Lame's constants, $\alpha$ is a thermal expansion constant, $\delta_{ij}$ is Kronecker's delta and $T$ is a temperature. The expression (7) presents the strain space thermoplasticity constitutive relation for isotropic materials and can be written in a general form as

$$d\sigma_{ij} = \begin{cases} C_{ijkl}(d\varepsilon_{kl} - \alpha\delta_{kl}dT) \quad \text{at} \quad F < 0 \quad \text{elasticity} \\ C_{ijkl}(d\varepsilon_{kl} - \alpha\delta_{kl}dT) - H\left(\frac{\partial F}{\partial \varepsilon_{kl}}d\varepsilon_{kl} + \frac{\partial F}{\partial T}dT\right)\frac{\partial F}{\partial \varepsilon_{ij}} \\ \qquad \text{at} \quad F = 0, \quad dF = \frac{\partial F}{\partial \varepsilon_{kl}}d\varepsilon_{kl} + \frac{\partial F}{\partial T}dT > 0 \quad \text{loading} \\ C_{ijkl}(d\varepsilon_{kl} - \alpha\delta_{kl}dT) \quad \text{at} \quad F = 0, \quad dF < 0 \quad \text{unloading} \end{cases} \tag{8}$$

For comparison the constitutive relations of classical thermoplasticity theory is also considered. It is well known that the total deformation may be presented as a sum of an elastic deformations, $d\varepsilon_{ij}^{e}$, plastic deformations, $d\varepsilon_{ij}^{p}$ and thermal deformations, $d\varepsilon_{ij}^{T}$[40], i. e.

$$d\varepsilon_{ij} = d\varepsilon_{ij}^{e} + d\varepsilon_{ij}^{p} + d\varepsilon_{ij}^{T} \tag{9}$$

where $d\varepsilon_{ij}^{e} = C_{ijkl}^{-1}d\sigma_{kl}$ and $d\varepsilon_{ij}^{T} = \alpha\delta_{ij}dT$. The plastic deformation is defined from the associated flow law as [14, 40]

$$d\varepsilon_{ij}^{p} = d\lambda_1 \frac{\partial f}{\partial \sigma_{ij}} \quad \text{at } f(\sigma_{ij}, T, \chi) = 0,\ df = \frac{\partial f}{\partial \sigma_{kl}}d\sigma_{kl} + \frac{\partial f}{\partial T}dT > 0 \tag{10}$$

where

$$\chi = \int \sigma_{ij} d\varepsilon_{ij}^{p}. \tag{11}$$

Using the consistency condition i. e. differentiate equation $f(\sigma_{ij}, T, \chi) = 0$ and taking into account the expression (11) and the associated flow law (10), one obtains

$$d\lambda_1 = h\left(\frac{\partial f}{\partial \sigma_{ij}}d\sigma_{ij} + \frac{\partial f}{\partial T}dT\right) \tag{12}$$

where

$$h = -\left(\frac{\partial f}{\partial \chi}\sigma_{ij}\frac{\partial f}{\partial \sigma_{ij}}\right)^{-1}. \tag{13}$$

Multiplying $\frac{\partial f}{\partial \sigma_{mn}}C_{mnij}$ to (9) and using (12) we obtain an expression for the scalar parameter which is expressed as

$$d\lambda_1 = \frac{\frac{\partial f}{\partial \sigma_{mn}}C_{mnij}(d\varepsilon_{ij} - d\varepsilon_{ij}^{T}) + \frac{\partial f}{\partial T}dT}{\frac{1}{h} + \frac{\partial f}{\partial \sigma_{ij}}C_{ijkl}\frac{\partial f}{\partial \sigma_{kl}}}. \tag{14}$$

Apply the Hooke's law to Eq.(9) yields

$$d\sigma_{ij} = C_{ijkl}\left(d\varepsilon_{kl} - d\lambda_1\frac{\partial f}{\partial \sigma_{kl}} - d\varepsilon_{kl}^{T}\right). \tag{15}$$

Substituting (14) into (15) gives the constitutive relation for the classical thermoplasticity theory i. e.,

$$d\sigma_{ij} = C_{ijkl}(d\varepsilon_{kl} - \alpha\delta_{kl}dT) - C_{ijkl}\frac{\frac{\partial f}{\partial \sigma_{mn}}C_{mnij}(d\varepsilon_{ij} - d\varepsilon_{ij}^T) + \frac{\partial f}{\partial T}dT}{\frac{1}{h} + \frac{\partial f}{\partial \sigma_{ij}}C_{ijkl}\frac{\partial f}{\partial \sigma_{kl}}}\frac{\partial f}{\partial \sigma_{kl}} \tag{16}$$

which may be rewritten in the following form

$$d\sigma_{ij} = \begin{cases} C_{ijkl}(d\varepsilon_{kl} - \alpha\delta_{kl}dT) \quad \text{at} \quad f < 0 \quad \text{elasticity} \\ C_{ijkl}(d\varepsilon_{kl} - \alpha\delta_{kl}dT) - C_{ijkl}\frac{\frac{\partial f}{\partial \sigma_{mn}}C_{mnij}(d\varepsilon_{ij} - d\varepsilon_{ij}^T) + \frac{\partial f}{\partial T}dT}{\frac{1}{h} + \frac{\partial f}{\partial \sigma_{ij}}C_{ijkl}\frac{\partial f}{\partial \sigma_{kl}}}\frac{\partial f}{\partial \sigma_{kl}} \\ \qquad \text{at} \quad f = 0, \quad df = \frac{\partial f}{\partial \sigma_{kl}}d\sigma_{kl} + \frac{\partial f}{\partial T}dT > 0 \quad \text{loading} \\ C_{ijkl}(d\varepsilon_{kl} - \alpha\delta_{kl}dT) \quad \text{at} \quad f = 0, \quad df < 0 \quad \text{unloading} \end{cases} \tag{17}$$

where $f(\sigma_{ij}, T, \chi) = 0$ is the loading surface in stress space. For a detail comparison of the thermoplasticity constitutive relations for strain space (8) and stress space (17), the loading functions $F(\varepsilon_{ij}, T, \omega)$ and $f(\sigma_{ij}, T, \chi)$ are chosen in a Mises's form, respectively i. e.

$$F = \frac{1}{2}e_{ij}e_{ij} - R(\omega, T) = 0 \tag{18}$$

$$f = \frac{1}{2}S_{ij}S_{ij} - K(\chi, T) = 0 \tag{19}$$

where $e_{ij}$ and $S_{ij}$ are deviators of the strain and stress tensors, $R$ and $K$ are hardening functions, $\omega$ and $\chi$ are plasticity parameters defined by the expressions (3) and (11), respectively. Substituting expressions (18) and (19) into (7) and (16) gives, respectively, the thermoplasticity constitutive relation for strain space

$$\begin{aligned} d\sigma_{ij} \quad = \quad & \lambda d\theta\delta_{ij} + 2\mu d\varepsilon_{ij} - (3\lambda + 2\mu)\alpha\delta_{ij}dT - H\left(\frac{\partial F}{\partial \varepsilon_{kl}}d\varepsilon_{kl} + \frac{\partial F}{\partial T}dT\right)e_{ij} \\ & \text{at} \quad F = 0 \quad \text{and} \quad dF = \frac{\partial F}{\partial \varepsilon_{kl}}d\varepsilon_{kl} + \frac{\partial F}{\partial T}dT > 0 \end{aligned} \tag{20}$$

where

$$H = -\left(2\frac{\partial R}{\partial \omega}\varepsilon_u^2\right)^{-1} \text{ and } \quad \varepsilon_u^2 = \frac{1}{2}e_{ij}e_{ij} \tag{21}$$

and the thermoplasticity constitutive relation for stress space

$$\begin{aligned} d\sigma_{ij} \quad = \quad & \lambda d\theta\delta_{ij} + 2\mu d\varepsilon_{ij} - (3\lambda + 2\mu)\alpha\delta_{ij}dT - \frac{4\mu^2 S_{ij}S_{kl}}{\frac{1}{h} + 4\mu\sigma_u^2}de_{kl} \\ & - \frac{2\mu S_{ij}}{\frac{1}{h} + 4\mu\sigma_u^2}\frac{\partial f}{\partial T}dT \quad \text{at} \quad f = 0 \quad \text{and} \quad df = \frac{\partial f}{\partial S_{ij}}dS_{ij} + \frac{\partial f}{\partial T}dT > 0 \end{aligned} \tag{22}$$

where

$$h = -\left(2\frac{\partial K}{\partial \chi}\sigma_u^2\right)^{-1} \text{ and } \quad \sigma_u^2 = \frac{1}{2}S_{ij}S_{ij}. \tag{23}$$

The notations $H$ and $h$ depend on hardening functions $R(\omega, T)$ and $K(\chi, T)$ which are defined from the experimental stress-strain curves constructed at different temperatures. Note that a set of stress-strain curves at different temperatures makes up the deformation surface for temperature dependent materials (Figure 1). The hardening functions $R(\omega, T)$ and $K(\chi, T)$ for strain space and stress space thermoplasticity theories, are obtained by generalizing the relations

$\varepsilon = \varepsilon(\sigma^p, T)$ and $\sigma = \sigma(\varepsilon^p, T)$ constructed on the base of experimental stress-strain curves at different temperatures (Figure 2). Note that the construction of the hardening functions $R(\omega), K(\chi)$ for strain and stress space plasticity theories are demonstrated in Nik Long et al. If the experimental deformation surface is approximated by two piecewise planes (Figure 1), the hardening functions for the strain and stress spaces constitutive relations, respectively, take the forms

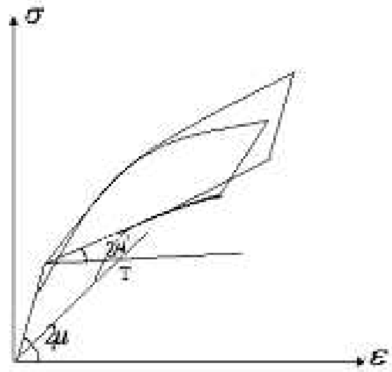


**Figure 1: Deformation Surface for Temperature Dependent and Stress $\boldsymbol{\sigma = \sigma(\varepsilon^p, T)}$ Space Theories**

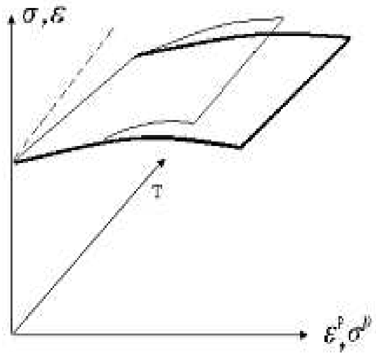


**Figure 2: Thermoplasticity Functions for Strain $\boldsymbol{\varepsilon = \varepsilon(\sigma^p, T)}$ Materials**

$$R(\omega, T) = \frac{\omega}{2(\mu - \mu')} + R^o + (T - T_0)\beta \tag{24}$$

$$K(\chi, T) = \frac{2\mu\mu'}{\mu - \mu'}\chi + K^o + (T - T_0)\gamma \tag{25}$$

where $\mu'$ is tangent modulus, and $\beta = \frac{\partial F}{\partial T}$ and $\gamma = \frac{\partial f}{\partial T}$ are the inclination angles to the temperature axis $OT$ in strain and stress spaces, respectively. Substituting Eqs. (24) and (25) into (20) and (22) gives the constitutive relations for strain and stress space thermoplasticity theories, respectively

$$d\sigma_{ij} = \lambda d\theta\delta_{ij} + 2\mu d\varepsilon_{ij} - (3\lambda + 2\mu)\alpha\delta_{ij}dT - \frac{\mu - \mu'}{\varepsilon_u^2}(e_{kl}de_{kl})e_{ij}$$

$$-\frac{\mu - \mu'}{\varepsilon_u^2}e_{ij}\frac{\partial F}{\partial T}dT \quad \text{at} \quad F = 0 \quad \text{and} \quad dF = e_{kl}de_{kl} + \frac{\partial F}{\partial T}dT > 0 \tag{26}$$

$$d\sigma_{ij} = \lambda d\theta\delta_{ij} + 2\mu d\varepsilon_{ij} - (3\lambda + 2\mu)\alpha\delta_{ij}dT - \frac{\mu - \mu'}{\sigma_u^2}(S_{kl}de_{kl})S_{ij}$$

$$-\frac{\mu - \mu'}{2\mu\sigma_u^2}S_{ij}\frac{\partial f}{\partial T}dT \quad \text{at} \quad f = 0 \quad \text{and} \quad df = S_{kl}dS_{kl} + \frac{\partial f}{\partial T}dT > 0 \tag{27}$$

The right hand side and the loading condition of (27) depend on the stress and strain deviators and temperature, whereas (26) depends only on strain tensor deviator and temperature. The dependence of the constitutive relations on strain tensors is convenience for formulation and numerical solution of the coupled thermoplasticity boundary value problems. It can be seen that the third term of the constitutive relations (26) and (27) are responsible for thermoelastic deformations, whereas the fifth is for thermoplastic deformations.

## 3. STRAIN SPACE THERMOPLASTICITY COSTITUTIVE RELATIONS FOR TRANSVERSELY ISOTROPIC MATERIALS

Mechanical properties of a large class of fiber reinforced, laminated composites and sedimentary rocks are markedly directional and may be considered as a transverse isotropy material. Let the axis $OX_3$ coincides with a transverse isotropy axis. The transversely isotropic tensor is invariant under the group of transformations generated by rotations about $OX_3$, symmetry with respect to a plane containing $OX_3$ and symmetry with respect to a plane perpendicular to $OX_3$ [4, 5]. The transversely isotropic strain tensors may be represented as a sum of four mutually orthogonal terms [4, 6]

$$\varepsilon_{ij} = \frac{\tilde{\theta}}{2}(\delta_{ij} - \delta_{i3}\delta_{j3}) + \varepsilon_{33}\delta_{i3}\delta_{j3} + p_{ij} + q_{ij}. \tag{28}$$

Note that the expansion (28) is similar to the expansion of the isotropic strain tensor to the spherical and deviator terms. In (28) $\varepsilon_{33}$, $\tilde{\theta}$ and $p_{ij}, q_{ij}$ plays the role of a spherical and deviator terms for transversely isotropic tensors, respectively, and are defined by the following expressions

$$p_{ij} = \varepsilon_{ij} + \frac{\tilde{\theta}}{2}(\delta_{i3}\delta_{j3} - \delta_{ij}) + \varepsilon_{33}\delta_{i3}\delta_{j3} - (\varepsilon_{i3}\delta_{j3} + \varepsilon_{j3}\delta_{i3}),$$
$$q_{ij} = \varepsilon_{i3}\delta_{j3} + \varepsilon_{j3}\delta_{i3} - 2\varepsilon_{33}\delta_{i3}\delta_{j3} \quad \text{and} \quad \tilde{\theta} = \varepsilon_{11} + \varepsilon_{22}.$$

Similarly, for the transversely isotropic stress tensor, the following expansion are valid

$$\sigma_{ij} = \tilde{\sigma}(\delta_{ij} - \delta_{i3}\delta_{j3}) + \sigma_{33}\delta_{i3}\delta_{j3} + P_{ij} + Q_{ij} \tag{29}$$

where

$$\begin{aligned} P_{ij} &= \sigma_{ij} + \tilde{\sigma}(\delta_{i3}\delta_{j3} - \delta_{ij}) + \sigma_{33}\delta_{i3}\delta_{j3} - (\sigma_{i3}\delta_{j3} + \sigma_{j3}\delta_{i3}), \\ Q_{ij} &= \sigma_{i3}\delta_{j3} + \sigma_{j3}\delta_{i3} - 2\sigma_{33}\delta_{i3}\delta_{j3} \quad \text{and} \quad \tilde{\sigma} = \frac{\sigma_{11} + \sigma_{22}}{2}. \end{aligned} \tag{30}$$

In (29), $\tilde{\sigma}, \sigma_{33}$ and $P_{ij}, Q_{ij}$ play the role of the spherical and deviatoric terms for the transversely isotropic stress tensors, respectively. According to the orthogonal expansions (28) and (29), similar to the loading surface in case of isotropy (2), we may introduce two loading functions for transversely isotropic materials [6, 33]

$$F_p(p_{ij}, T, \omega_p) = 0 \tag{31}$$

$$F_q(q_{ij}, T, \omega_q) = 0 \tag{32}$$

where $\omega_p = \int p_{ij} dP_{ij}^p$ and $\omega_p = \int q_{ij} dQ_{ij}^p$ are the plasticity deformation parameters, $P_{ij}^p$ and $Q_{ij}^p$ are the stress relaxation tensors. Using Eqs.(31) and (32), from Eq.(1) we obtain

$$d\sigma_{ij}^p = dP_{ij}^p + dQ_{ij}^p \tag{33}$$

where

$$dP_{ij}^{p} = d\tilde{\lambda}_p \frac{\partial F_p}{\partial p_{ij}} \quad \text{at} \quad F_p = 0 \quad \text{and} \quad \frac{\partial F_p}{\partial p_{kl}} dp_{kl} + \frac{\partial F_p}{\partial T} dT > 0,$$
$$dQ_{ij}^{p} = d\tilde{\lambda}_q \frac{\partial F_q}{\partial q_{ij}} \quad \text{at} \quad F_q = 0 \quad \text{and} \quad \frac{\partial F_q}{\partial q_{kl}} dq_{kl} + \frac{\partial F_q}{\partial T} dT > 0,$$

$d\tilde{\lambda}_p$ and $d\tilde{\lambda}_q$ are the scalar differential parameters. In case of transversely isotropy, Eq.(7) takes the form

$$d\sigma_{ij} = C_{ijkl}(d\varepsilon_{kl} - \alpha_{kl} dT) - d\tilde{\lambda}_p \frac{\partial F_p}{\partial p_{ij}} - d\tilde{\lambda}_q \frac{\partial F_q}{\partial q_{ij}} \tag{34}$$

where

$$\begin{aligned} C_{ijkl} &= \lambda_1 \gamma_{ij}\gamma_{kl} + \lambda_4(\gamma_{ik}\gamma_{jl} + \gamma_{il}\gamma_{jk}) + \lambda_3 \delta_{i3}\delta_{j3}\delta_{k3}\delta_{l3} + \lambda_2(\delta_{i3}\delta_{j3}\delta_{kl} + \delta_{k3}\delta_{l3}\delta_{ij}) \\ &\quad + \lambda_5\left(\delta_{il}\delta_{k3}\delta_{j3} + \delta_{i3}\delta_{l3}\delta_{kj} + \delta_{ik}\delta_{l3}\delta_{j3} + \delta_{jl}\delta_{k3}\delta_{i3}\right), \\ \gamma_{ij} &= \delta_{ij} - \delta_{i3}\delta_{j3}, \quad \alpha_{kl} = \alpha_1(\delta_{kl} - \delta_{i3}\delta_{j3}) + \alpha_3 \delta_{i3}\delta_{j3} \end{aligned}$$

$\alpha_1, \alpha_3$ are the thermal expansion constants, $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ and $\lambda_5$ are the elastic constants for transversely isotropic material [6]. Using the consistency condition, and taking into account Eqs.(31−35), we obtain

$$d\tilde{\lambda}_p = H_p\left(\frac{\partial F_p}{\partial p_{ij}} dp_{ij} + \frac{\partial F_p}{\partial T} dT\right), \tag{35}$$

$$d\tilde{\lambda}_q = H_p\left(\frac{\partial F_q}{\partial q_{ij}} dq_{ij} + \frac{\partial F_q}{\partial T} dT\right) \tag{36}$$

where

$$H_p = -\left(\frac{\partial F_p}{\partial \omega_p} p_{ij} \frac{\partial F_p}{\partial p_{ij}}\right)^{-1}, \quad H_q = -\left(\frac{\partial F_q}{\partial \omega_q} q_{ij} \frac{\partial F_q}{\partial q_{ij}}\right)^{-1}. \tag{37}$$

Substituting expressions (35) and (36) into (34) gives the strain space thermoplasticity constitutive relations for transversely isotropic materials i. e.

$$d\sigma_{ij} = \begin{cases} C_{ijkl}(d\varepsilon_{kl} - \alpha_{kl} dT) \quad \text{at} \quad F_p < 0, \quad F_q < 0 \quad \text{elasticity} \\ C_{ijkl}(d\varepsilon_{kl} - \alpha_{kl} dT) - H_p\left(\frac{\partial F_p}{\partial p_{kl}} dp_{kl} + \frac{\partial F_p}{\partial T} dT\right) \frac{\partial F_p}{\partial p_{kl}} \\ \quad -H_q\left(\frac{\partial F_q}{\partial q_{kl}} dq_{kl} + \frac{\partial F_q}{\partial T} dT\right) \frac{\partial F_q}{\partial q_{kl}} \quad \text{loading} \\ \quad \text{at} \quad F_p = 0, \quad \frac{\partial F_p}{\partial p_{kl}} dp_{kl} + \frac{\partial F_p}{\partial T} dT > 0 \\ \quad \text{at} \quad F_q = 0, \quad \frac{\partial F_q}{\partial q_{kl}} dq_{kl} + \frac{\partial F_q}{\partial T} dT > 0 \\ C_{ijkl}(d\varepsilon_{kl} - \alpha_{kl} dT) \quad \text{at} \quad F_p = 0, \quad F_q = 0 \\ \quad \text{and} \quad dF_p < 0, dF_q < 0 \quad \text{unloading} \end{cases} \tag{38}$$

Expression (38) represents the strain space thermoplasticity constitutive relation for transversely isotropic materials.

## 4. STRESS SPACE THERMOPLASTICITY CONSTITUTIVE RELATION FOR TRANSVERSELY ISOTROPIC MATERIALS

It is well known that the classical plasticity theories are based on the associated flow law (10) with loading functions considered in the stress space. According to the orthogonal expansion (29) for transversely isotopic tensors, we may consider two loading surfaces $f_p$ and $f_q$ for stress space thermoplasticity theories

$$f_p(P_{ij}, T, \chi_p) = 0, \chi_p = \int P_{ij} dp_{ij}^p \tag{40}$$

$$f_q(Q, T, \chi_q) = 0, \chi_q = \int Q_{ij} dq_{ij}^p \tag{41}$$

where $P_{ij}, Q_{ij}$ are the stress tensors of the orthogonal expansion (29-30); $\chi_p, \chi_q$ are the plasticity parameters similar to (11), $dp_{ij}^p, dq_{ij}^p$ are the plastic deformations defined from the associated flow law (10). Using Eqs.(9) and (10) and taking into account consistency conditions, apply Hook's law, and after some manipulations by analogy to (15), we can find the stress space thermoplasticity constitutive relation for transversely isotropic materials

$$d\sigma_{ij} = \begin{cases} C_{ijkl}(d\varepsilon_{kl} - \alpha_{kl} dT) \quad \text{at} \quad f_p < 0, \quad f_q < 0; \text{ elasticity,} \\ C_{ijkl}(d\varepsilon_{kl} - \alpha_{kl} dT) - C_{ijkl} \dfrac{\frac{\partial f_p}{\partial P_{mn}} C_{mnkl} dp_{kl} + \frac{\partial f_p}{\partial T} dT}{\frac{1}{h_p} + \frac{\partial f_p}{\partial P_{ij}} C_{ijkl} \frac{\partial f_p}{\partial P_{kl}}} \dfrac{\partial f_p}{\partial P_{ij}} \\ \qquad - C_{ijkl} \dfrac{\frac{\partial f_q}{\partial Q_{mn}} C_{mnkl} dq_{kl} + \frac{\partial f_q}{\partial T} dT}{\frac{1}{h_q} + \frac{\partial f_q}{\partial Q_{ij}} C_{ijkl} \frac{\partial f_q}{\partial Q_{kl}}} \dfrac{\partial f_q}{\partial Q_{kl}} \\ \qquad \text{at} \quad f_p = 0, \quad df_p = \dfrac{\partial f_p}{\partial P_{ij}} dP_{ij} + \dfrac{\partial f_p}{\partial T} dT > 0 \\ \qquad \text{at} \quad f_q = 0, \quad df_q = \dfrac{\partial f_q}{\partial Q_{ij}} dQ_{ij} + \dfrac{\partial f_q}{\partial T} dT > 0; \text{ loading} \\ C_{ijkl}(d\varepsilon_{kl} - \alpha_{kl} dT) \quad \text{at} \quad f_p = 0, \quad df_p < 0, \\ \qquad \qquad \text{at} \quad f_q = 0, \quad df_q < 0; \text{ unloading} \end{cases} \tag{42}$$

where

$$h_p = -\left(\frac{\partial f_p}{\partial \chi_p} P_{ij} \frac{\partial f_p}{\partial P_{ij}}\right)^{-1}, \quad h_q = -\left(\frac{\partial f_q}{\partial \chi_q} Q_{ij} \frac{\partial f_q}{\partial Q_{ij}}\right)^{-1}. \tag{43}$$

The right hand side of (38) depends on strain tensors $(p_{ij}, q_{ij})$ and its increments $(dp_{kl}, dq_{kl}, d\varepsilon_{kl})$and temperature, whereas the right hand side of (42) depends on stress tensors $(P_{ij}, Q_{ij})$ and strain tensor increments $(dp_{kl}, dq_{kl}, d\varepsilon_{kl})$ and temperature. Comparison of two constitutive relations shows that the dependence of(38) on strain tensor and its increments and temperature, allows to formulate the thermoplasticity boundary problems for displacements and temperature. Note that in case of stress space constitutive relation (42), the boundary value problem depends on stress tensor besides the strain tensor and temperature. It can be seen from the constitutive relations (38) and (43) that in loading case, that

the first terms are responsible for the thermoelastic deformations, the second terms of the numerators of second and third terms for the thermoplastic deformations, whereas others for the plastic deformations. Note that the thermoplastic

terms in constitutive relations (38) and (42) describe the joint influence of the thermal and mechanical factors to the deformation process. Considering the aforementioned thermoplasticity constitutive relations for isotropic or transversely isotropic materials in conjunction with a motion equation and heat equations may be formulated a coupled thermomechanical boundary value problems with a corresponding initial and boundary conditions. Note that all equations of the coupled themoplasticity boundary value problem should be written for displacement and temperature increments. In [26], using the strain space thermoplasticity constitutive relation for isotropic materials (26) the dynamic one dimensional coupled thermoplasticity problem is numerically solved, and by comparison with the known results the validity of the applied constitutive relations is verified. If the thermal terms are neglected, Eqs. (38) and (42) give the strain and stress constitutive relations for plastic transversely isotropic materials [33].

## 5. CONCLUSIONS

The thermoplastic constitutive relations for isotropic and transversely isotropic materials with a loading surfaces in stress and strain spaces are considered. In case of the strain space thermoplasticity theories, the alternative associated flow law are used, which is based on the stress relaxation tensor and states that the increment of the stress relaxation tensor is orthogonal to the loading surface considered in the strain space. We note that the value of the stress relaxation tensor is equal to the product of the plastic deformation by the elastic constants. Using the classical and alternative associated flow laws the stress and strain space thermoplasticity constitutive relations for transversely isotropic materials are constructed and compared. The formulation of the coupled thermoplasticity boundary value problems based on the stress and strain space constitutive relations are discussed. It is found that the strain space thermoplastic constitutive relations are more convenient for the formulation and numerical solution of the coupled thermoplasticity boundary value problems than in stress space thermoplasticity theories.

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